Dual discount rates

For project net present value calculations

20th December 2000 – Updated 29 September 2004

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Summary

In the course of his work in oil and gas reservoir engineering, the author has made an examination of the choice of objectives in the economic optimisation of oil-field development. As a result of this work, a new method of calculating project net present value (NPV) has been derived from the Capital Asset Pricing Model that in its simplest form is as follows:-

1. Remove the assumption that the project discount rate can be assumed to be constant in time.
2. Assume that the project cash-flow can be divided into a number of components (e.g. revenue and expenditure), each of which can be considered to have constant risk and, hence, a constant discount rate.
3. Evaluate the present value of each constant-risk component with its own discount rate (e.g. 20% per annum for revenue, 6% for expenditure) and then calculate the overall project NPV as the sum of the present values of the constant-risk components of the cash-flow.

Such a method, which can be called a "dual discount" or "multi-discount" method, resolves or lessens a number of anomalies with the conventional single discount rate NPV calculation method.

Conventional calculations of project NPV give rise to a number of anomalies and questions including -

- Should the discount rate be the cost of capital or the opportunity cost of capital?
- A high NPV:NPC ratio (NPC = net present value of capex) can occur in a project with a very low internal rate of return and vice-versa.
- Conventional discounting appears to understate the costs or benefits of long time-frame events, such as the abandonment of oil-field installations or changes in environmental quality.
- Oil price risk increases with time, so acceleration can reduce oil price risk, but conventional models do not capture this.

To address these anomalies, a dual discount NPV model was derived mathematically from the Capital Asset Pricing Model (CAPM), by three different routes

- applying the CAPM to each moment in time to create a difference/differential equation and solving it (noting that project risk and hence project discount rate varies with time);
- by a simple argument based on the additivity of NPV;
- by looking at the effects, on an oil-field development, of pre-selling all of the hydrocarbons in the futures' markets.

The resultant NPV is a measure of the value of an investment opportunity, given its non-diversifiable risk e.g. (for an oil-field project) oil and gas price risk. It will usually be significantly lower than the NPV calculated conventionally using a cost-of-capital discount rate. The dual discount NPV should correspond to the price that the investment opportunity should fetch in an idealised, open market (noting that a project with a dual discount NPV of zero is a good, ordinary investment for the CAPEX it takes).

The dual discount NPV method may be particularly significant for oil companies in that

- it suggests that most fields would better be developed with significantly more wells (up to 50%) than is customary at the moment;
- this might add to add 5 - 20% to the value of these fields, and hence a significant percentage to the value of the oil company as a whole.

The method was first described in the public domain in the UK patent application GB2366409A, dated 2nd Sept 2000 and available on http://gb.espacenet.com.

Problems with conventional economic criteria

There are at least five major unsolved problems associated with the conventional methods for evaluating project economics -

- Should the discount rate be the cost of capital or the opportunity cost of capital?
- A high NPV:NPC ratio (NPC = net present value of capex) can occur in a project with a very low IRR and vice-versa.
- IRR (internal rate of return) is not always well defined - a project with more than one tranche of capital expenditure might appear to have two IRR e.g. 12% and 26%.
- For oil field development projects, oil price risk increases with time, so acceleration can reduce oil price risk, but conventional models do not capture this.
- Conventional discounting appears to understate the costs or benefits of long time-frame events, such as the abandonment of oil-field installations or changes in environmental quality.

NPV is a measure of value, but it depends critically on the choice of discount rate. In the oil industry, there has been a great deal of discussion as to whether it was better to use the cost of capital (e.g. the interest rate to be paid on loans or the expected return to shareholders, or some combination - see Sinha and Poole 1987) or the opportunity cost of capital (the return expected from alternative investment opportunities).

It is straightforward to construct a mathematical argument for using the opportunity cost of capital in order to maximise the overall company rate of return, but it is usual now to ignore this argument and use the weighted average cost of capital. Both sides in the argument do not address the paradox that capital is usually the one thing that does not need to be discounted, since it is spent at the beginning of the project. The discount rate affects OPEX and revenue. Implicit in conventional discounting is the assumption that post-tax revenue converts itself back into capital and so should be discounted at the rates appropriate for capital. This assumption is fine *unless* your discount rate includes some allowance for uncertainty .

If one uses the weighted average cost of capital, then one is usually faced with an excess of investment opportunities, and so there is the need for a capital efficiency measure, usually NPV/NPC, where NPC is the net present value of the capital employed. (In contrast, the opportunity cost of capital method does not require such a hurdle - all projects with NPV greater or equal to zero go ahead, since the discount rate is precisely the rate of return of the most marginal project).

NPV/NPC would be the appropriate criteria to use if one was faced with one-off problem "How do I allocate my capital to maximise NPV". The real problem is "How do I allocate my capital now, and next year and the year after that etc in order to maximise the long-term value of my company." Since projects return the capital they use, often over differing time frames, NPV/NPC is not always a very useful measure.

For example, NPV/NPC does not distinguish between a two-month project with an NPV/NPC of 0.1 (very attractive) and a twenty-year project with an NPV/NPC of 0.1 (usually unattractive). Indeed, it is possible to construct example cash-flows that have any desired combination of NPV/NPC and rate of return (providing that NPV/NPC > 0 and the rate of return is greater than the discount rate). So one could create an example with a very low rate of return and a very high NPV/NPC or with a very high rate of return and a very low NPV/NPC. (If d is the discount rate, V is the desired example NPV/NPC and r is the desired rate of return, then a project with a cash-flow of

$$-1 \text{ at time } 0$$

$$+(1+r)^{\ln(V+1)/[\ln(1+r)-\ln(1+d)]} \text{ at time } \ln(V+1)/[\ln(1+r)-\ln(1+d)]$$

has the desired properties).

While internal rate of return or IRR is a very useful concept, it also has problems associated with it. In particular, there is the problem that the formal definition of IRR (the discount rate at which the project has zero NPV) can lead to the situation in which a project with more than one tranche of capital expenditure might appear to have two IRR e.g. 12% and 26%. This problem is probably one of the big reasons why attention shifted from rate of return more on to NPV.

With regards to the choice of discount rate, the problem with IRR contributed to the use of the cost of capital model, rather than the opportunity cost model, since the latter requires an IRR calculation (to determine the opportunity cost).

There are also problems in conventional project economics with the treatment of uncertainty. It is usual to set up the evaluation of NPV in a deterministic way and then vary the parameters to get the range of possible NPVs. For most parameters, except oil and gas price, it is usual to concentrate on the expectation NPV. It is important, however, to remember that the value of a project depends not only on the expectation NPV, but also on the degree of risk - the variance in our range of NPV values.

In our oil field development example, oil and gas price uncertainty is usually treated differently, since it can affect all of an oil company's fields simultaneously. It is usual for oil companies to use screening criteria to control this risk - e.g. insisting that the all field development plans achieve a positive NPV at an oil price of $11/bbl and a discount rate of 10%. As far as the author knows, these criteria are pragmatically derived, rather than the result of much theory.

A key problem with this treatment of uncertainty is that it ignores the fact that oil and gas price uncertainty increases with time. This can be seen by examination of historical price data and by the fact that it is conventional to model the oil price as a random walk. One consequence of way that oil price uncertainty increases with time is that technical measures (acceleration of production profiles) can then help reduce oil price uncertainty. The benefits of this (any reduction in uncertainty has benefits) are ignored by conventional economic analysis.

Another problem is that conventional discounting appears to understate the costs or benefits of long time-frame events, such as the abandonment of oil-field installations or changes in environmental quality. As a pragmatic solution to this problem, some oil companies have started (2004) to use separate, lower discount rates for abandonment expenditure. In environmental area (Tol 2003), there is also a move to the use of dual discounting, interpreted to be because of either

- savings and emission control having different pure rates of time preference;
- changes in the marginal willingness to pay for environmental quality.

It is reasonable to conclude from this list of problems that there are weaknesses in the basic theory that underlies the conventional approach to project economics. Hence, it is worthwhile to start back at the beginning and examine whether one of the basic assumptions used to build up the theory of project finance is questionable. This has been done, and the results are a slightly new model for project economics, called the "dual discount" method.

The dual discount NPV model

The dual discount NPV can be summarised as an NPV model that uses two discount rates. The expenditure stream is discounted at the cost of capital (e.g. 6% - 8%). The revenue stream is discounted as a rate that takes account of oil and gas price risk (e.g. 20%).

A slightly more elaborate form, to take account of the effects of taxes and costs that might vary with oil price is, firstly, to express the predicted post-tax cash-flow in the form

Cash-flow for year i = $a_i.P_i + b_i$

where P_i is the oil-price in year i, and a_i and b_i are the appropriate linear coefficients.

Then the net present value of this cashflow is

$$\mathrm{NPV} = \sum_{t=1}^{n}\left(\frac{a_t.P_0}{\left(1+E(\mathbf{r_p})\right)^t} + \frac{b_t}{\left(1+r_f\right)^t}\right)$$

where P_0 is starting-point oil price, and $E(\mathbf{r}_p)$ is the expected rate of return on writing oil futures.

Deriving the dual discount NPV model from the Capital Asset Pricing Model

The dual discount NPV model can be derived mathematically from the Capital Asset Pricing Model (CAPM), by three different routes

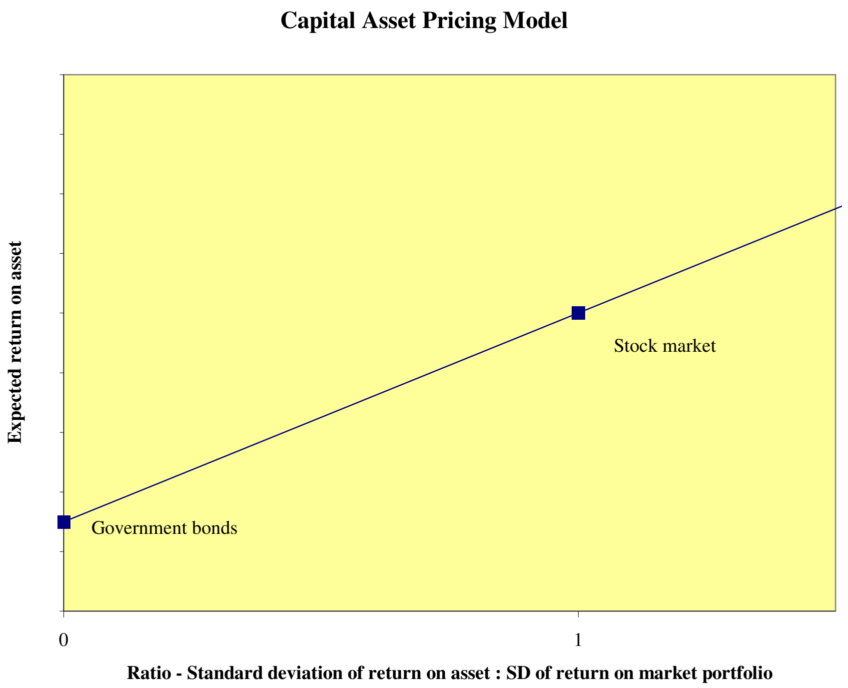
- applying the CAPM to each moment in time to create a difference/differential equation and solving it (noting that project risk and hence project discount rate varies with time);
- by a simple argument based on the linearity of NPV;
- by looking at the effects of pre-selling all of the hydrocarbons in the futures' markets.

The key difference in assumptions between the dual discount NPV model and conventional project finance models is that the dual discount NPV model does not make the assumption that project risk and project discount rate are constant through-out the life of the project.

The Capital Asset Pricing Model (CAPM) states that

- the expected rate of return on a capital asset is a linear function of its non-diversifiable risk (measured by the covariance of the asset's rate of return and the rate of return from a diversified portfolio of shares divided by the variance of the rate of return from the diversified portfolio)
- the value of the capital asset is determined by this relationship.

This can be visualised as follows (making the (false) simplification that the covariance / variance term equals the ratio of the standard deviations).



CAPM is an extremely powerful theory that has many immediate implications for the evaluation of project economics -

- The value of a project depends very much on its uncertainty. This implies that it is undesirable to follow the conventional approach of creating a model that deals with uncertainty by looking at a range of NPVs, each calculated under the assumption of certainty. Instead, it is better to model the uncertainty *prior* to calculating the NPV
- However many oil price scenarios there are, a given field development scheme has only ever one NPV - this NPV should reflect all the different scenarios and their probabilities.

The steps in the mathematical derivation of the dual discount NPV model from CAPM were as follows -

1. It was noticed that the CAPM equation refers only to a single time period. Hence, it was decided to apply to single time periods in the project, rather than to the project as a whole. This is equivalent to removing the assumption that the project has a single discount rate that applies across its entire life. This assumption is, in any case, very questionable since it implies that the project has constant risk throughout its life.

2. Project risk was split into oil/gas price risk (non-diversifiable) and other risks (diversifiable).

3. Oil-price risk was modelled as a random walk.

4. The difference/differential equations were solved to give the dual discount equation.

The full mathematical derivation is given in Appendix 1.

The dual discount equation was initially created from this argument. However, after one becomes familiar with the basic idea of using two discount rates, then some simpler arguments suggest themselves.

The first simple argument is one based on the additivity of NPV - the principle that states that

NPV(Cashflow 1 + Cashflow 2) = NPV(Cashflow 1) + NPV(Cashflow 2)

It would be straightforward to split a project into an expenditure cashflow and a revenue cashflow (after a little fiddling to split up tax effects etc). If this were done, then each cashflow could be evaluated separately. The expenditure cashflow would then have nothing to do with oil and gas price risk, and so could be evaluated with a discount rate that reflects the fact that all of the risk is diversifiable. The revenue cashflow contains the oil and gas price risk, which can be reflected in some standard way in the discount rate.

This additivity argument can probably be made mathematically rigorous. However, it is almost too simple to be convincing, by itself, for such a change in the way NPV is evaluated. One might well ask "If it is so simple, why has no-one thought of it before?"

The experience of the author might provide a partial answer to this question. The author has worked on this problem for several years, and yet he never thought of using two discount rates simultaneously. He did recognise that CAPM applies to an instant in time, so he did consider whether project risk and project discount rate might vary with time. But the very strength of CAPM, with its clear-cut justification of a single project discount rate at any one time, would have further prevented the author from considering the possibility of using two discount rates. It was the process of solving the CAPM-derived equations that showed that the effects of the time-dependent single discount rate could be expressed in terms of two time-independent discount rates.

A consequence of the additivity argument is that one is not necessarily restricted to just two discount rates - the cash-flow might be split into three or more parts, each capturing some risk with its own discount rate.

A third argument, particularly relevant to oil-field development projects can be put forward to justify the dual discount NPV method. Consider the act of pre-selling oil or gas on the futures market. This is a zero NPV transaction. (In reality, there are some small transaction costs, but these can be ignored). A fair market price is paid for the reduction of risk.

Consider lumping together a oil-field development scheme with the act of selling all of its oil and gas on the futures market. By the additivity of NPV,

NPV (Field development + selling the oil and gas on the futures' market) =

NPV(Field development) + NPV(Selling the oil and gas on the futures' market) =

NPV(Field development)

Hence, by using futures' prices, one can evaluate a project NPV that takes into account oil and gas price uncertainty.

Comparing dual discount NPV with conventional models

It may be useful at this point to look at how dual discount NPV would differ from conventional NPV.

Dual discount NPV is a measure of value, given uncertainty. In contrast, conventional NPV is a measure of value, given certainty. With conventional NPV, uncertainty is added afterwards, in the form of different scenarios. With Dual discount NPV, the modelling of the crucial uncertainties should be done at the beginning, and their effects should be incorporated into one or more of the discount rates used.

A project with a Dual discount NPV of zero would be a perfectly adequate investment, given the risks. A project with a conventional NPV would not be a good investment.

Dual discount NPV needs a more complicated treatment of abandonment than the one used in conventional NPV calculations. Work is ongoing in this area.

It is hoped, in the long-term, the use of dual discount NPV will reduce the need for NPV/NPC criteria, by reducing the gap between the owner's valuation of an undeveloped field and the price that it might fetch if sold. (If the two matched, one would not be faced with a CAPEX shortage, since one could sell surplus opportunities).

In the shorter term, dual discount NPV should probably be used with a (NPC x payback) measure of capital requirements. Hence, the corporate ranking criteria would be the Dual discount NPV : (NPC x payback) ratio.

Dual discount NPV resolves the *cost of capital* vs *opportunity cost of capital* problem. The cost of capital is used for the expenditure discount rate. The overall project discount rate (which varies with time) will be close to the opportunity cost of capital.

Dual discount NPV also points out a way to solve the problem of multiple IRRs. The rate of return of a project is an instantaneous quantity, which varies during project life. The concept of a constant project IRR is erroneous, so it is not surprising that the attempt to define such a quantity gives rise to paradoxes.

The derivation of dual (or multiple) discount rates presented here also helps support the use that has been made of dual discount rates for evaluating improvements in environmental quality and (in a totally separate area) oil field abandonment costs. However, the derivation suggests that the theoretical justification for low discount rates for environmental quality may lie predominantly in a possible negative correlation between the rate of return for environmental quality and the rate of return for conventional economic investments. Such a negative correlation might represent an example of the extrapolation of the capital asset pricing model to below the risk-free interest rate – an extrapolation that is often discussed as a theoretical possibility, but one not encountered amongst conventional economic investments.

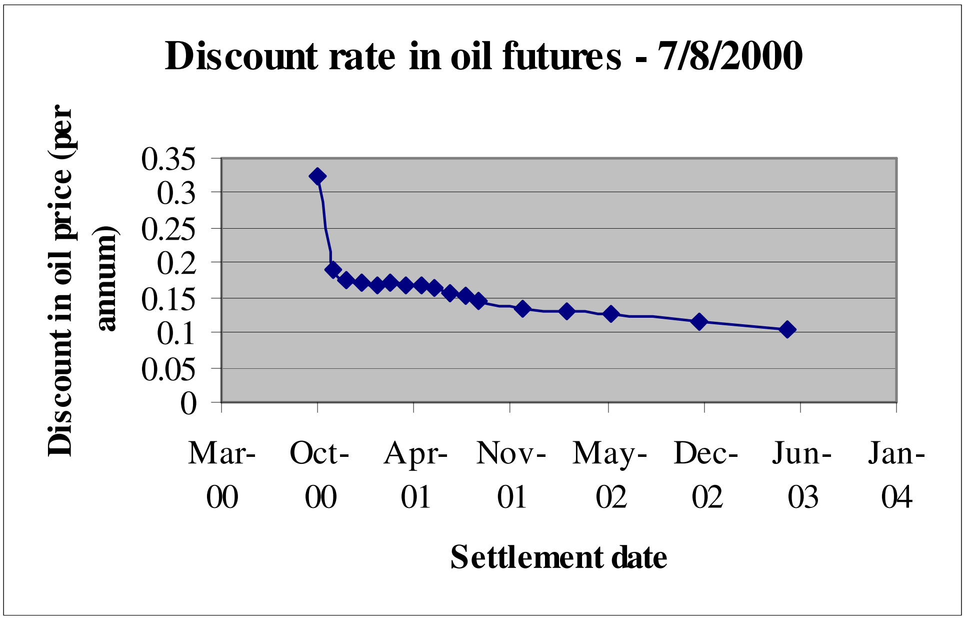
Such a justification for a low discount rate for environmental quality might be explained qualitatively as follows.

- If economic growth was certain, it might make sense to delay environmental improvements until we can more easily pay for them, in line with calculations from conventional single discount rate calculations.
- However, the uncertainty in future economic growth makes it worthwhile to implement environmental improvements sooner rather than later. If there was a major collapse of the economy (because of political chaos, war, a meteorite strike or whatever), it would be all the more important to have a high quality environment, to make the most of the enforced simplicity of our lives in the new, poorer world, or as a basis on which to rebuild.

Calibrating the revenue discount rate with futures' data

Broadly speaking, what has been established so far is that oil and gas price risk should be captured in the oil and gas price discounting. There remains the question as to what should be the discount rate and whether it might vary with time. In the mathematical derivation of the dual discount equation, it was assumed, for simplicity, that the oil and gas discount rates would be constant. However, there is a source of objective information about the value of future oil production – namely the oil futures market. The author has not had access to the large volume of detailed work done in this area, only to some price data. Hence, what follows is only very preliminary analysis. The author would be very grateful for more information about this subject.

An examination was made of the way the offered futures oil price declined with the length of the futures' contract. Data from 7th Aug 2000 is shown in the plot below.

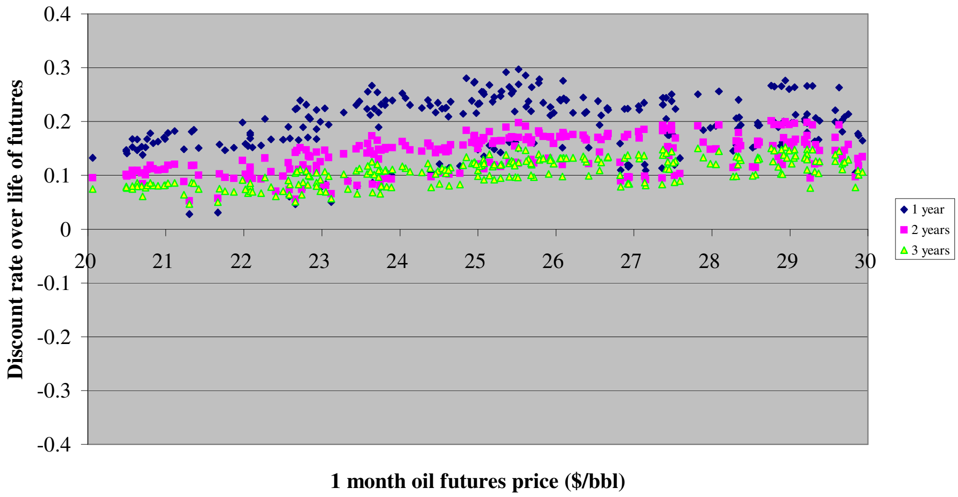


This suggested that it would be reasonable to discount future oil prices with 12% per annum. When this is added to the 7% discount for time effect, this gives a combined discount rate of 20%.

(The calculation is (1 + 0.12) x (1 + 0.07) = (1 + 0.20)).

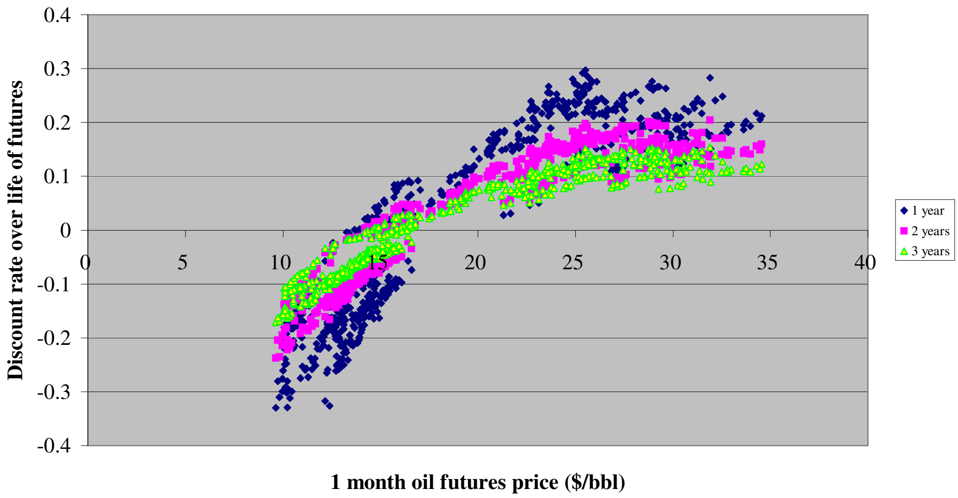
Examination of the full January 1998 - September 2000 data set from the International Petroleum Exchange, London, broadly supports a 12% price discount rate, for the 20 - 30 US$ /bbl oil price starting range, as can be seen in the plot overleaf.

Discount rates from IPE oil futures 1998 - 2000



However, there appears to be a noticeable decline in discount rates with time. This would be consistent with the oil price following a random walk that tended to converge to some long-term equilibrium price (as opposed to the neutral random walk model of oil price that is used in Appendix 1). Examination of the data over the full price range adds further evidence for this and suggests that the long-term equilibrium price is possibly $16 - $17 / bbl.

Discount rates from IPE oil futures 1998 - 2000



Although this does not affect the basic conclusion of the dual discount NPV model - that oil price uncertainty should be captured in revenue cashflow discounting - it suggests that more work is needed to determine how to carry out this discounting for time periods much longer than 3 years (the period over which the futures' market works).

References

Tol, R.S.J. (2003) On dual-rate discounting. Economic Modelling 21 (2003) pp. 95-98 (http://www.uni-hamburg.de/Wiss/FB/15/Sustainability/ecmoddualdiscount.pdf)

Sinha M.K., Poole A.V. (1987) Selection of a discount rate or minimum acceptable IRR. Society of Petroleum Engineers Paper 16843 (www.spe.org)

Appendix 1 - Mathematical derivation of the dual discount NPV method

It is necessary, first, to establish some notation.

Notation

The proof requires the use of random variables (e.g. the range of possible oil prices for next year), samples from these random variables (e.g. when calculating the covariance of next year's oil price and the return on a market portfolio) and actual values (e.g. the current oil price). Random variables will take bold type. Samples from the random variable will use -[i] notation. Actual values will take normal type.

P_j	Actual oil price at time j
$\mathbf{P_{j+1}}$	The oil price at time j+1 as a random variable
$\mathbf{P_{j+1}} \vert P_j$	The random variable for the oil price at time j+1, given that the oil price at time j was Pj
$(\mathbf{P_{j+1}} \vert P_j)[i]$	Sample [i] of the above random variable
$E(\mathbf{P_{j+1}} \vert P_j)$	The expected value of the price at time j+1, given that the oil price at time j was Pj
$p(\mathbf{P_{j+1}}[i])$	The probability that the oil price at time j+1 is $\mathbf{P_{j+1}}[i]$.
r_f	The risk-free rate of return

$\mathbf{r_m}$	The rate of return on a portfolio of stocks as a random variable
$\mathbf{r_p}$	The rate of return on oil futures as a random variable
V_j	The value of the remaining future cash-flow of the project at time j. (NB - At time j, the future cash-flow is a random variable, but value of the future cash-flow is an actual number - what it can be bought or sold for at time j. Prior to time j, the value at time j is a random variable).
$\mathbf{r_j}$	The rate of return of the project over the time between t = j-1 and t = j as a random variable.
$\mathbf{V_{j+1}} \mid P_j$	The random variable for the value of the remaining future cash-flow of the project at time j+1, given that the oil-price at time j is P_j.
$Cov(\mathbf{r_p}, \mathbf{r_m})$	The covariance of $\mathbf{r_m}$ and $\mathbf{r_p}$
FP(j,1)	The price at time j of a one time-period oil future - a single unit of oil, with delivery at time j+1

Assumptions

1. The project cash-flow in time-period t can be expressed as $a_t.P_t + b_t$ where a_t and b_t are known constants. (In reality, they can be random variables, providing they are independent of $\mathbf{r_m}$. In our oil-field example, a_t and b_t would depend on the geology of the field, the development scheme chosen and all the costs. To a first approximation, these can be considered to be independent of the FT100 or the Dow Jones stock market indices).

2. All the cash-flow in a time-period occurs at the end of the time period.
3. The risk-free rate of return, r_f , and the expected rate of return on a market portfolio, $E(\mathbf{r_m})$, are constant.
4. The Capital Asset Pricing Model (CAPM) applies, i.e. the value of an asset is such that, over a single time period,

$$E(\mathbf{r}) = r_f + \frac{Cov(\mathbf{r}, \mathbf{r_m})}{Var(\mathbf{r_m})}\left(E(\mathbf{r_m}) - r_f\right)$$

5. The oil price follows a random walk such that it is equally likely to go up as to go down in price. More precisely,
   $E(\mathbf{P_{j+1}} | P_j) = P_j$
6. The expected rate of return on oil futures, $E(\mathbf{r_p})$, is constant. Equivalently, the financial risk β on oil futures is constant.

Theorem

The value at time j of the future cash-flow of a project is given by the expression

$$V_j = \sum_{t=j+1}^{n}\left(\frac{a_t . P_j}{\left(1 + E(\mathbf{r_p})\right)^{t-j}} + \frac{b_t}{\left(1 + r_f\right)^{t-j}}\right)$$

where n is the last year of the project.

Proof

The proof is by induction on j, working downwards. We start by noting that the expression to be proved is equivalent to

$$V_j = \sum_{t=j+1}^{n+1}\left(\frac{a_t . P_j}{\left(1 + E(\mathbf{r_p})\right)^{t-j}} + \frac{b_t}{\left(1 + r_f\right)^{t-j}}\right) \qquad (1)$$

where $a_{n+1} = b_{n+1} = 0$.

To start the induction on this second formulation, we note equation (1) is trivially true for j = n + 1, since

0 = 0 + 0

We will proceed to show that if equation (1) is true for j + 1, then it is true for j.

Now, by the Capital Asset Pricing Model, CAPM, the value, at time j, of the future cash-flow, is given by the expression

V_j = (Expected value of time-period (j+1) cash-flow + expected value at time (j+1) of the future cash-flow) all discounted at the appropriate expected rate of return, $E(\mathbf{r_{j+1}}|P_j)$

$$= \frac{a_{j+1}.E(\mathbf{P_{j+1}}|P_j) + b_{j+1} + E(\mathbf{V_{j+1}}|P_j)}{1 + E(\mathbf{r_{j+1}}|P_j)} \qquad (2)$$

Using assumption (5) about oil price expectations, this can be re-arranged to give

$$E(\mathbf{r_{j+1}}|P_j) = \frac{a_{j+1}.P_j + b_{j+1} + E(\mathbf{V_{j+1}}|P_j)}{V_j} - 1 \qquad (3)$$

At this point in the proof, most of these terms are unknown. However, with the help of additional equations, it will be possible to derive their values.

CAPM also states that the expected rate of return is a function of the degree of non-diversifiable risk, as given by the expression

$$E(\mathbf{r_{j+1}}|P_j) = r_f + \frac{Cov(\mathbf{r_{j+1}}|P_j, \mathbf{r_m})}{Var(\mathbf{r_m})}.(E(\mathbf{r_m}) - r_f) \qquad (4)$$

Our main inductive statement contains enough information for us to express this in terms of the expected return on buying oil futures, as captured in the following lemma

Lemma

If

$$V_{j+1} = \sum_{t=j+2}^{n+1} \left(\frac{a_t.P_{j+1}}{(1+E(\mathbf{r_p}))^{t-(j+1)}} + \frac{b_t}{(1+r_f)^{t-(j+1)}} \right)$$

(i.e. the main inductive statement is true for j+1) then

$$E\left(\mathbf{V_{j+1}}\middle|P_j\right) = P_j\left(\sum_{t=j+2}^{n+1}\frac{a_t}{\left(1+E(\mathbf{r_p})\right)^{t-(j+1)}}\right) + \sum_{t=j+2}^{n+1}\frac{b_t}{\left(1+r_f\right)^{t-(j+1)}} \quad (5)$$

and

$$\frac{Cov\left(\mathbf{r_{j+1}}\middle|P_j, \mathbf{r_m}\right)}{Var\left(\mathbf{r_m}\right)}.\left(E\left(\mathbf{r_m}\right) - r_f\right) = \frac{P_j}{V_j}\left(\sum_{t=j+1}^{n+1}\frac{a_t}{\left(1+E(\mathbf{r_p})\right)^{t-j}}\right).\left(E(\mathbf{r_p}) - r_f\right) \quad (6)$$

For a proof of the lemma, see later. Note that the proof of the lemma does not depend on any results from the main theorem.

Using the lemma and the assumption that the main inductive statement, equation (1), is true for j+1, we can equate the two expression for $E(\mathbf{r_{j+1}}|P_j)$ given in equations (3) and (4) to get

$$r_f + \frac{P_j}{V_j}\left(\sum_{t=j+1}^{n+1}\frac{a_t}{\left(1+E(\mathbf{r_p})\right)^{t-j}}\right).\left(E(\mathbf{r_p}) - r_f\right) = \frac{a_{j+1}.P_j + b_{j+1} + E\left(\mathbf{V_{j+1}}\middle|P_j\right)}{V_j} - 1 \quad (7)$$

Re-arranging and using equation (5), we get

$$V_j = \frac{1}{\left(1+r_f\right)}.\left(\begin{array}{l} P_j.a_{j+1} + b_{j+1} + P_j\left(\sum_{t=j+2}^{n+1}\frac{a_t}{\left(1+E(\mathbf{r_p})\right)^{t-(j+1)}}\right) + \\ \sum_{t=j+2}^{n+1}\frac{b_t}{\left(1+r_f\right)^{t-(j+1)}} - P_j\left(\sum_{t=j+1}^{n+1}\frac{a_t}{\left(1+E(\mathbf{r_p})\right)^{t-j}}\right).\left(E(\mathbf{r_p}) - r_f\right)\end{array}\right) \quad (8)$$

Using the fact that

$$\frac{1}{1-r_f}.\left(1 - \frac{E(\mathbf{r_p}) - r_f}{1+E(\mathbf{r_p})}\right) = \frac{1}{1+E(\mathbf{r_p})} \quad (9)$$

equation (8) can be simplified to

$$V_j = \sum_{t=j+1}^{n+1}\left(\frac{a_t.P_j}{\left(1+E(\mathbf{r_p})\right)^{t-j}} + \frac{b_t}{\left(1+r_f\right)^{t-j}}\right) \quad (1)$$

i.e. If equation (1) is true for j+1, then it is true for j. As discussed, equation (1) is trivially true for j = n + 1. Hence, by induction, equation (1) is true for all j less than or equal to n+1.

QED

Proof of Lemma

Consider the probability distribution of the future value of the project in a year's time, $\mathbf{V_{j+1}}$. To work with this random variable, it is useful to consider individual values taken by the random variable, $\mathbf{V_{j+1}}[i]$. By the assumption in the lemma

$$\mathbf{V_{j+1}}[i] = \sum_{t=j+2}^{n+1} \left(\frac{a_t.\mathbf{P_{j+1}}[i]}{\left(1+E(\mathbf{r_p})\right)^{t-(j+1)}} + \frac{b_t}{\left(1+r_f\right)^{t-(j+1)}} \right)$$

From this, we can calculate the expected value of $\mathbf{V_{j+1}}$ given the current oil price, P_j.

$$E\left(\mathbf{V_{j+1}} \middle| P_j\right) = \sum_i \left(p\left(\mathbf{P_{j+1}}[i] \middle| P_j\right) . \left(\sum_{t=j+2}^{n+1} \frac{a_t.\mathbf{P_{j+1}}[i]}{\left(1+E(\mathbf{r_p})\right)^{t-(j+1)}} + \sum_{t=j+2}^{n+1} \frac{b_t}{\left(1+r_f\right)^{t-(j+1)}} \right) \right)$$

$$= \left(\sum_{t=j+2}^{n+1} \frac{a_t}{\left(1+E(\mathbf{r_p})\right)^{t-(j+1)}} \right) . \left(\sum_i p\left(\mathbf{P_{j+1}}[i] \middle| P_j\right) . \mathbf{P_{j+1}}[i] \right) + \sum_{t=j+2}^{n+1} \frac{b_t}{\left(1+r_f\right)^{t-(j+1)}}$$

Since $\sum_i p\left(\mathbf{P_{j+1}}[i] \middle| P_j\right) . \mathbf{P_{j+1}}[i]$ is the expected oil price at time $j+1$, given the oil price at time j, then, by assumption 5, this is equal to Pj. Hence,

$$E\left(\mathbf{V_{j+1}} \middle| P_j\right) = P_j \left(\sum_{t=j+2}^{n+1} \frac{a_t}{\left(1+E(\mathbf{r_p})\right)^{t-(j+1)}} \right) + \sum_{t=j+2}^{n+1} \frac{b_t}{\left(1+r_f\right)^{t-(j+1)}}$$

i.e. Equation (5) is proved.

The next stage is to prove equation (6). The first step is to note that

$$\frac{Cov\left(\mathbf{r_{j+1}} \middle| P_j, \mathbf{r_m}\right)}{Var\left(\mathbf{r_m}\right)} . \left(E\left(\mathbf{r_m}\right) - r_f\right) = \frac{Cov\left(\mathbf{r_{j+1}} \middle| P_j, \mathbf{r_m}\right)}{Cov\left(\mathbf{r_p}, \mathbf{r_m}\right)} . \frac{Cov\left(\mathbf{r_p}, \mathbf{r_m}\right)}{Var\left(\mathbf{r_m}\right)} . \left(E\left(\mathbf{r_m}\right) - r_f\right) \tag{11}$$

Since CAPM applies also to $\mathbf{r_p}$, equation (11) can be converted to

$$\frac{Cov\left(\mathbf{r_{j+1}} \middle| P_j, \mathbf{r_m}\right)}{Var\left(\mathbf{r_m}\right)} . \left(E\left(\mathbf{r_m}\right) - r_f\right) = \frac{Cov\left(\mathbf{r_{j+1}} \middle| P_j, \mathbf{r_m}\right)}{Cov\left(\mathbf{r_p}, \mathbf{r_m}\right)} . \left(E\left(\mathbf{r_p}\right) - r_f\right) \tag{12}$$

We will show that both covariances in equation (12) can be expressed as multiples of $Cov(\mathbf{P_{j+1}} | P_j, \mathbf{r_m})$.

Firstly, let us deal with Cov($\mathbf{r_p}$, $\mathbf{r_m}$).

Let FP(j,1) be the price at time j of a one time-period oil future - a single unit of oil, with delivery after one time period. (Futures are usually quoted in terms of an agreed price at the time of delivery. Here, we want to deal with the current price - this is the price at time of delivery discounted at the risk-free rate of return, assuming that there are no risk of either side defaulting on the contract).

At the time of delivery, the unit of oil will be worth P_{j+1}. Hence, the actual return on the oil future will be

$$r_p = \frac{P_{j+1}}{FP(j,1)} - 1 \qquad (13)$$

and the expected return will be

$$E(\mathbf{r_p}) = \frac{E(\mathbf{P_{j+1}} | P_j)}{FP(j,1)} - 1 \qquad (14)$$

Solving equation 14 for FP(j,1) and substituting into equation 13 gives

$$r_p = \frac{P_{j+1}.(1 + E(\mathbf{r_p}))}{P_j} - 1 \qquad (15)$$

Hence, at time j, when the future value P_{j+1} is unknown, the distribution of r_p over the next time period (from t = j to t = j+1) is given by

$$\mathbf{r_p} = \mathbf{r_p} | P_j = \frac{\mathbf{P_{j+1}} | P_j.(1 + E(\mathbf{r_p}))}{P_j} - 1 \qquad (16)$$

Consequently, if we evaluate Cov($\mathbf{r_p}$, $\mathbf{r_m}$) at time j, we get

$$Cov(\mathbf{r_p}, \mathbf{r_m}) = Cov(\mathbf{P_{j+1}} | P_j, \mathbf{r_m}) \frac{1 + E(\mathbf{r_p})}{P_j} \qquad (17)$$

Moving on to the other covariance term in the right-hand side of equation (12), Cov($\mathbf{r_{j+1}} | P_j$,$\mathbf{r_m}$), we proceed in the same manner - calculating the actual rate of return achieved, r_{j+1}, given an actual oil price at time j+1 and then deducing the probability distribution of r_{j+1} at time j.

From the definition of rate of return, and assumptions 1 and 2

r_{j+1} = (cash-flow during the time-period [j,j+1] + value of remaining project at time j+1) / (value of remaining project at time j)

$$r_{j+1} = \frac{a_{j+1}.P_{j+1} + b_{j+1} + V_{j+1}}{V_j} \tag{18}$$

Using the assumption contained in the statement of the lemma to expand V_{j+1} gives

$$r_{j+1} = \frac{a_{j+1}.P_{j+1} + b_{j+1} + \sum_{t=j+2}^{n+1} \frac{a_t.P_{j+1}}{(1+E(\mathbf{r_p}))^{t-(j+1)}} + \sum_{t=j+2}^{n+1} \frac{b_t}{(1+r_f)^{t-(j+1)}}}{V_j}$$

$$= \frac{P_{j+1} \sum_{t=j+1}^{n+1} \frac{a_t}{(1+E(\mathbf{r_p}))^{t-(j+1)}} + \sum_{t=j+1}^{n+1} \frac{b_t}{(1+r_f)^{t-(j+1)}}}{V_j} \tag{19}$$

Looking at the distribution of r_{j+1} at time j, we get

$$\mathbf{r_{j+1}} \big| P_j = \frac{\mathbf{P_{j+1}} \big| P_j \sum_{t=j+1}^{n+1} \frac{a_t}{(1+E(\mathbf{r_p}))^{t-(j+1)}} + \sum_{t=j+1}^{n+1} \frac{b_t}{(1+r_f)^{t-(j+1)}}}{V_j} \tag{20}$$

Hence

$$\mathrm{Cov}(\mathbf{r_{j+1}} \big| P_j, \mathbf{r_m}) = \frac{1}{V_j} \left(\mathrm{Cov}(\mathbf{P_{j+1}} \big| P_j, r_m) \sum_{t=j+1}^{n+1} \frac{a_t}{(1+E(\mathbf{r_p}))^{t-(j+1)}} \right) \tag{21}$$

Substituting equations (17) and (21) into the covariance terms in equation (12) and cancelling the $\mathrm{Cov}(\mathbf{P_{j+1}} | P_j, \mathbf{r_m})$ terms gives

$$\frac{\mathrm{Cov}\left(\mathbf{r_{j+1}}\middle|P_j,\mathbf{r_m}\right)}{\mathrm{Var}(\mathbf{r_m})}.\left(E(\mathbf{r_m})-r_f\right)=\frac{\frac{1}{V_j}\sum_{t=j+1}^{n+1}\frac{a_t}{\left(1+E(\mathbf{r_p})\right)^{t-(j+1)}}}{\left(\left(1+E(\mathbf{r_p})\right)\Big/P_j\right)}.\left(E(\mathbf{r_p})-r_f\right)$$

$$=\frac{P_j}{V_j}\left(\sum_{t=j+1}^{n+1}\frac{a_t}{\left(1+E(\mathbf{r_p})\right)^{t-j}}\right).\left(E(\mathbf{r_p})-r_f\right)$$

This proves equation (6), the final statement in the lemma and concludes the formal derivation of the dual discount rate model.

Appendix 2 - Economic objectives and oil-field development design

The choice of economic criteria has a big effect on oil-field development design, since it drives the decision about what is the optimal speed at which to develop the field i.e. what is the optimal number of wells to drill; what is the optimal facilities size. Of particular importance is the fact that the two alternative ways of rationing capital - using a NPV/NPC hurdle or increasing the discount rate - have different effects on development decisions. Using an NPV/NPC hurdle leads one to drill fewer wells and to develop the field slower. Using a higher discount rate leads one to drill more wells, and so spend more CAPEX, but this is compensated by a faster payback.

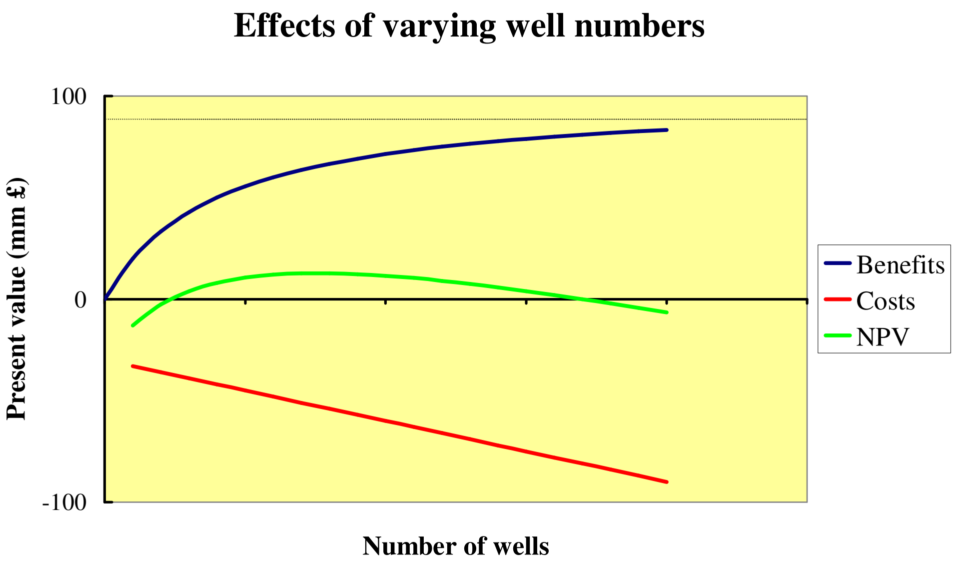
To understand all these issues, it is useful to consider what are the effects of increasing the number of wells drilled on a field.

The first effect of increasing the number of wells in a field development plan is to increase costs. Assuming that the facilities are scaled to the number of wells, both opex and capex are likely to be approximately a linear function of the number of wells in the plan. In consequence, the present value of the total costs will also be approximately a linear function of the number of wells, of the form a + b.N, where

- a is the fixed expenditure necessary irrespective of the number of wells - e.g. much of pipeline construction costs or the baseline facilities costs
- N is the number of wells drilled
- b represents the costs that vary with the number of wells - this obviously includes the direct costs of drilling the wells, but also includes the extra costs incurred in facilities etc.

On the benefits side, increasing the number of wells speeds up the field production, so you get your oil faster. This obviously increases the present value of the production. Increasing the number of wells also usually increases reserves and ultimate recovery (although there are circumstances where the opposite can happen, such as in a water-flood where increases in throughput rate can damage the displacement efficiency).

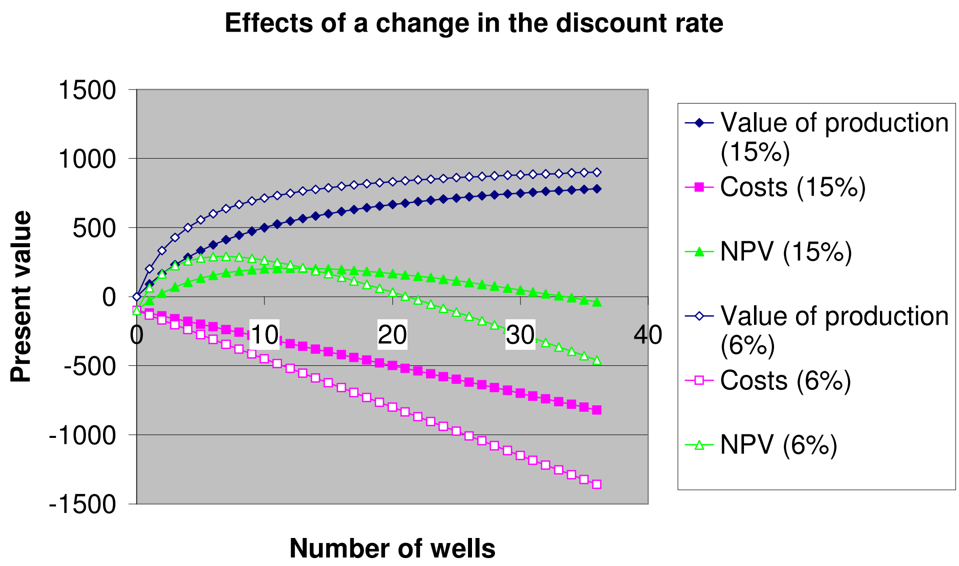
The benefits of speeding up production and the increases in reserves are both subject to diminishing returns and will always be less than the value of having all the available oil for sale now. Using this value as an asymptote, and the fact that the benefit of no wells is zero, we can construct a benefit vs well numbers relationship that will, qualitatively, look like the relationship in the plot below.



Adding together the present value of the benefits and the present value of the costs (a negative quantity) gives the NPV. It can be seen that the straight line of costs and the diminishing returns on the benefits side give rise to an NPV function that rises to a maximum and then declines.

If, as is usual, there is a shortage of capital and a NPV/NPC hurdle is used, then the number of wells is usually chosen so that the last well added to the plan just meets the hurdle - i.e. (extra NPV added by this well) / (extra NPC required) = hurdle. This equates to choosing a point on the NPV curve to the left of the peak.

If the shortage of capital is dealt with by increasing the discount rate, then the effect of this is to squash the revenue curve to the right and to make the costs curve shallower. The results of this are to lower the calculated NPVs substantially and, very importantly, to increase the number of wells to be drilled. This is illustrated in the plot below.



This is the effect using conventional project economics.

The dual discount NPV method is significant for oil companies in that

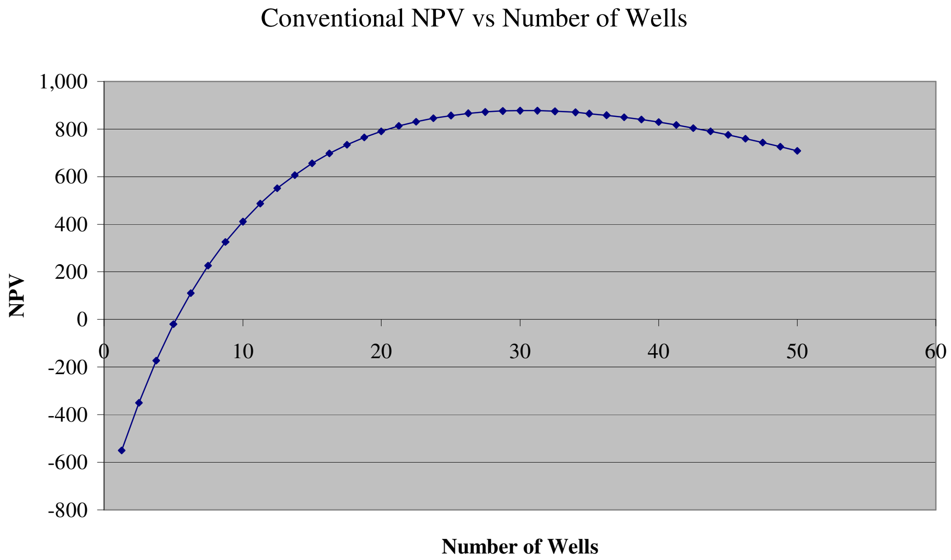
- it suggests that most fields would better be developed with 30% to 50% more wells than is customary at the moment;
- this might add to add 5 - 20% to the value of these fields, and hence a significant percentage to the value of the company as a whole.

This is illustrated by the following simplified example.

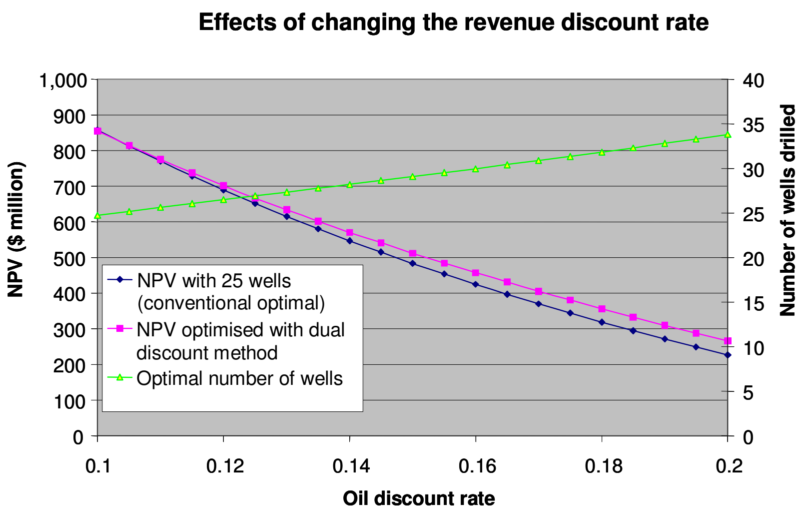
Consider a field with the following characteristics

- 200 million barrels oil reserves (assumed independent of well numbers);
- 1 million barrels/year initial well rates;
- 20% of reserves are produced in plateau, then exponential decline sets in;
- $30 million extra CAPEX costs needed per extra well (e.g. well costs, costs of larger facilities, costs of increase in platform size etc);
- $800 million fixed CAPEX (independent of number of wells);
- $20 / barrel Year 1 oil price;
- production ramps up over first year
- tax effects and OPEX are ignored for this example (OPEX might be considered to be contained in the CAPEX costs).

Conventional project economics (10% discounting, with a 0.3 NPV/NPC hurdle) suggests that the relationship between NPV and well numbers is as in the following plot, and that the optimal number of wells to drill is 25, giving an NPV of $857 million (at $20 / barrel oil price).



With the Dual discount NPV method, it may not be known for certain what should be the correct revenue discount rate, so it is useful to look at a range of possible values. As the revenue discount rate is gradually increased to 20% and the NPV/NPC hurdle is phased out, the change in the calculated optimal NPV is illustrated in the following plot, together with the NPV that would result from going ahead with the 25 well development (i.e. the development optimised under current project economics).



It can be seen from this plot that

1. If the correct revenue discount rate is 20%, then the use of conventional economics leads to a 25 well development with an NPV of $226 million, while the optimal is a 34 well development with an NPV of $266 million - i.e. the true value of the project is increased by 17% by using the dual discount NPV method.
2. Conventional project economics calculates an NPV (not designed to be the market price of the development opportunity) very much higher

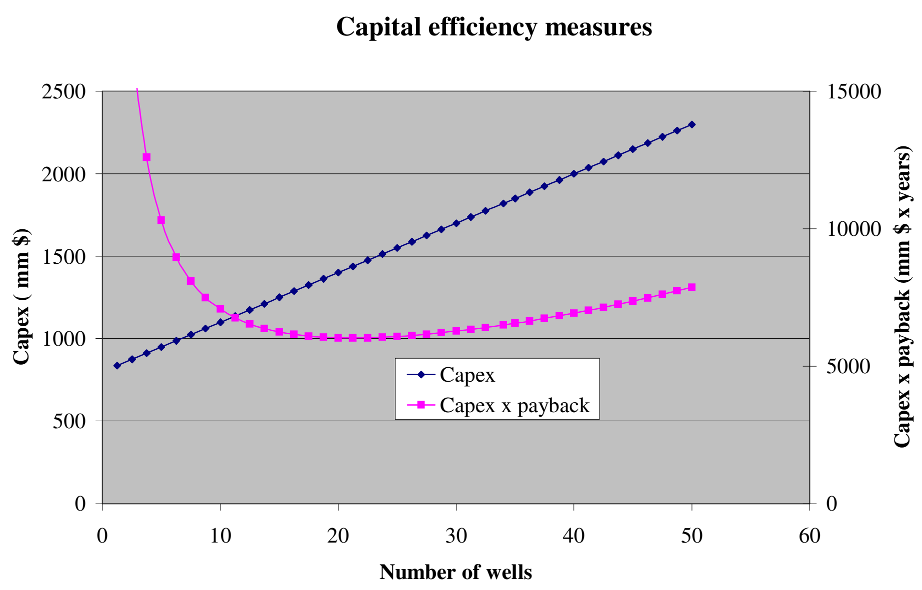
than the dual discount NPV (designed to be an estimate of the market price that the development opportunity might fetch). This is important, because it is possible that the over-estimation of value has contributed to the lack of a good market for undeveloped oil fields - and so to the problem of capital constraints (one is not faced with a capital constraint if it is possible realise the value of assets by selling them instead of developing them).

One might raise the question "Maybe the extra value comes from the removal of the NPV/NPC criteria?" The answer to this is that the NPV/NPC criteria are not very helpful in comparing options that involve acceleration. A development scheme with more wells may use more CAPEX, but it uses it for a shorter time. In theory, the expenditure discount rate should reflect all the costs of the capital, and so do away with the need for further capital efficiency measures. In practice (while the theory is being developed and confirmed), one may want to use an empirical measure of the amount of capital used and the length of time that it is used for.

Hence, it is proposed that a better measure of the amount of capital used (rather than just NPC) is

(CAPEX employed) x Payback.

A comparison, for this example, of the two measures is given in the plot below.



It can be seen that (CAPEX x payback) does not increase much on going from 25 wells to 34 wells. Indeed, the ratio of dual discount NPV to (CAPEX x payback) is higher (0.041) for the 34 well development than for the 25 well development (0.037). The comparison of the two options is summarised in the table below

US$ million

Number of wells	Conventional NPV (10% disc)	Dual discount NPV (20% rev disc)	CAPEX	CAPEX x payback	(Dual discount NPV) : (CAPEX x payback) ratio
25	857	226	1,550	6,078	0.037
34	870	266	1,820	6,498	0.041